Problem 3.

"Composition of Borel measurable functions is Borel measurable"

Proof. It suffices to show that, if

$$\forall t \in \mathbb{R}, \quad f^{-1}((-\infty, t]) = \{x \in \mathbb{R} : f(x) \leq t\} \text{ is Borel,}$$

then

$$\text{for all Borel sets } E \subset \mathbb{R}, \quad f^{-1}(E) \text{ is Borel.}$$

But in fact a general Borel set $E \subset \mathbb{R}$ can be constructed by taking countable unions, intersections, and complements of sets of the form $(-\infty, t]$.

Explicitly: by taking countable unions, the collection $\{(-\infty, t] : t \in \mathbb{R}\}$ generates any open interval of the form (∞, t_2). By taking $(\infty, t_2) \cap (-\infty, t_1]^c$, we can then get any open interval (t_1, t_2). The structure theorem on open sets in $\mathbb{R}$ states that any open set in $\mathbb{R}$ is a countable disjoint union of open intervals, so we can construct any open set in $\mathbb{R}$. Then by the definition of Borel set, we can generate all the Borel sets by taking countable unions, intersections, and complements of open sets.

Suppose that E is generated by taking countable unions, intersections, and complements of the sets $\{(0, t_j] : j \in \mathbb{N}\}$. Then $f^{-1}(E)$ will be expressible as by taking the corresponding countable unions, intersections, and complements of $\{f^{-1}((0, t_j]) : j \in \mathbb{N}\}$. But the latter collection $\{f^{-1}((0, t_j]) : j \in \mathbb{N}\}$ contains only Borel sets, by hypothesis, and so the result can only be Borel. □

Problem 4.

"Any $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere equal to a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$"

Proof. First we prove Lusin on all of $\mathbb{R}$. (This may possibly be overkill.)

Fix some small $\eta > 0$. Apply the finite-measure version of Lusin's theorem to each of the intervals $I_k = [k + \frac{\eta}{2^{|k|}}, k + 1 - \frac{\eta}{2^{|k|}}]$ for $k \in \mathbb{Z}$, setting $\epsilon_k = \frac{\eta}{2^{|k|}}$ in each interval, to find continuous functions $g_k : \mathbb{R} \to \mathbb{R}$ such that on some $E'_k \subset I_k$ with $m(I_k - E'_k) < \epsilon_k$ such that $f|_{E'_k} = g_k$.

Let $E' = \bigcup_{k=-\infty}^{\infty} E'_k$. Note that $m(\mathbb{R} - E') < 3\eta \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|}} = 9\eta$.

Now each E'_k may be chosen so as to be compact (closed and bounded), and therefore has a maximum and a minimum element, denoted A_k and a_k respectively. (One way of seeing this is that the identity function, like any other continuous function, assumes a maximum and a minimum on a compact set.) Then join the point $(A_k, g_k(A_k))$ to $(a_{k+1}, g_{k+1}(a_{k+1}))$ by a straight line segment for each $k \in \mathbb{Z}$. Then we can stitch together $g_k|_{E'_k}$ and $g_{k+1}|_{E'_{k+1}}$ continuously.

Thus for any $\eta > 0$ we can create a continuous function $G : \mathbb{R} \to \mathbb{R}$ for which $f|_{E')} = G$ and $m(\mathbb{R} - E') < 9\eta$, which can be made arbitrarily small. □

Now to the main problem. The (pointwise) limits of sequences of continuous functions are Borel measurable. To show this, we can exactly imitate the proof that pointwise limits of measurable functions are measurable. See Royden ch.3 Prop 9 (p.60-61) if you didn't catch this in class.

By Lusin on all of $\mathbb{R}$, we can get a sequence of continuous functions $G^{(1)}, G^{(2)}, G^{(3)}, ...$ such that for each $j \in \mathbb{N}$, $f = G^{(j)}$ except on a set of measure at most $\frac{1}{j}$ (just pick $\eta = \frac{1}{9j}$). Take $G : \mathbb{R} \to \mathbb{R}$ to be the Borel function defined by $G(x) := \lim_{j\to\infty} G^{(j)}(x)$.

For any point $x \in \mathbb{R}$, note that $G^{(j)}(x) \to f(x)$ as $j \to \infty$, **except** if x is in the j-th bad set $\mathbb{R} - (E')^{(j)}$ for infinitely many $j \in \mathbb{N}$. So the set of all $x \in \mathbb{R}$ for which $G^{(j)}(x)$ does **not** converge pointwise to $f(x)$ has measure bounded above by $m(\mathbb{R} - (E')^{(j)}) = \frac{1}{j}$ for all $j \in \mathbb{N}$, and therefore is of measure zero. Thus $f(x) = G(x)$ almost everywhere. $\square$